

ELECTRICAL ANALOG SIMULATION OF PROBLEMS OF ENERGY AND MASS TRANSFER

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The analog simulation of problems of energy and mass transfer on two RC networks with various boundary conditions is described. Solutions are presented for one- and two-dimensional problems, together with a solution for contact problems.

The analytic solution of problems of energy and mass transfer with complex initial and boundary conditions often presents insuperable mathematical difficulties, since these problems reduce to mathematical models, involving complex second-order partial differential equations of parabolic type. These equations, together with boundary conditions of various kinds, have been solved for certain important particular cases by A. V. Luikov and his school [1].

The present article is devoted to the electrical analog simulation of differential equations of energy and mass transfer of the type

$$\frac{\partial t}{\partial \tau} = a_q \nabla^2 t + \frac{\epsilon r c_m}{c_q} \frac{\partial \theta}{\partial \tau},$$

$$\frac{\partial \theta}{\partial \tau} = a_m \nabla^2 \theta + a_m \delta \theta \nabla^2 t \quad (1)$$

using RC networks.

The electrical model was constructed on the basis of the MN-7 analog computer.

§1. We first present the solution of the one-dimensional problem. The electrical circuit is shown in Fig. 1. On network I-I we measured the voltage v , the

analog of the temperature t ; on network II-II the voltage w , the analog of the mass transfer potential θ . If we write Eqs. (1) in dimensionless coordinates [3], the system takes the form

$$\frac{\partial T}{\partial Fo} = \nabla^2 T + Ko^* \frac{\partial \Theta}{\partial Fo},$$

$$\frac{\partial \Theta}{\partial Fo} = Lu \nabla^2 \Theta + Lu Pn \nabla^2 T. \quad (2)$$

The resistances R_1 and R_2 and the capacitances C_1 , C_2 were selected so as to ensure the equality of the Fourier, Luikov, Kossovich, and Posnov numbers:

$$Fo = \frac{a_q \tau}{l^2} = \frac{\tau_e}{R_1(C_1 + C) l_e^2},$$

$$Lu = \frac{a_m}{a_q} = \frac{R_1(C_1 + C)}{R_2(C_2 + C) [1 - C^2/(C_1 + C)(C_2 + C)]},$$

$$Ko^* = \epsilon r c_m \theta^*/c_q t^* = C w^*/(C_1 + C) v^*,$$

$$Pn = \delta_0 t^*/\theta^* = C R_2 v^*/(C_1 + C) R_1 w^*. \quad (3)$$

The boundary conditions are as follows: at the lower edge of the model

$$v = v^* = \text{const}, \quad \frac{\partial w}{\partial x} = 0$$

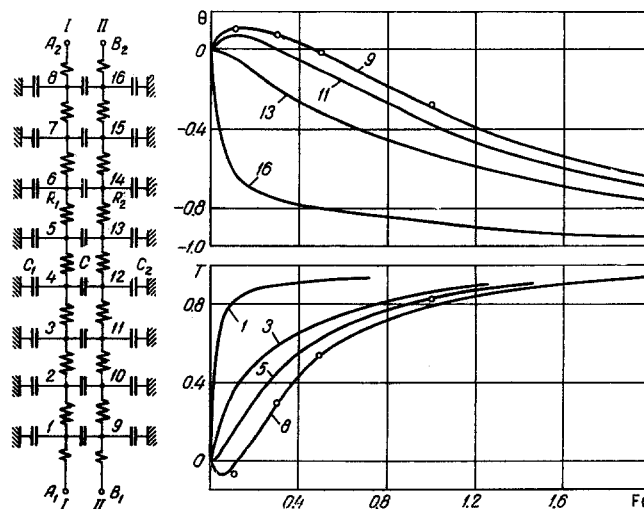


Fig. 1. Circuit of electrical model and graphs of T and Θ vs. Fo at various points of the medium in the one-dimensional problem.

$$\left(T(0; Fo) = 1, \frac{\partial \Theta(0; Fo)}{\partial N} = 0 \right), \quad (4)$$

at the upper edge of the model

$$w = w^* = \text{const}, \quad \frac{\partial v}{\partial x} = 0$$

$$\left(\Theta(1; Fo) = -1, \frac{\partial T(1; Fo)}{\partial N} = 0 \right). \quad (5)$$

The initial conditions are

$$T(X; 0) = 0, \quad \Theta(X; 0) = 0, \quad (6)$$

which corresponds to zero charge on all the capacitor plates. The experiment was carried out by supplying an electric potential $v^* = 100$ V to the point A_1 and a potential $w^* = -100$ V to the point B_2 at time $\tau_e = 0$. The time dependence of the relative temperature T and the mass transfer potential Θ is shown in Fig. 1 for various points of the medium: the electric potential $v = f(\tau_e)$, simulating the temperature T , was measured at points 1, 3, 5, and 8 of network I-I, and $w = f(\tau_e)$, simulating the mass transfer potential, was measured at points 9, 11, 13, and 16 of network II-II. It is clear from the graph $\Theta = f(Fo)$ that all the curves asymptotically approach $\Theta = -1$, while all the $T = f(Fo)$ curves asymptotically approach $T = 1$. The closer the point of network I-I to the boundary point A_1 , the more rapid the transient process and, conversely, the closer the point of network II-II to the point B_1 , the more prolonged the transient process.

We will consider the variation of $\Theta = f(Fo)$ at point 9. At first, the potential Θ increases, reaching a maximum at $Fo = 0.15$, although a negative potential $\Theta = -1$ was connected at the boundary of network II-II. This behavior of the curve is attributable to the fact that at the corresponding point 1 of network I-I the derivative $\partial T / \partial Fo$ takes a large value, and since in the neighborhood of point 9 at small Fo , $\nabla^2 \Theta$ is relatively small as a result of the remoteness of point 9 from point B_2 (where $\Theta = -1$), at small Fo the derivative $\partial \Theta / \partial Fo$ takes positive values. If $\nabla^2 \Theta$ were equal to zero, then

$$\frac{\partial \Theta}{\partial Fo} = \frac{Lu Pn}{1 + Lu Pn Ko^*} \frac{\partial T}{\partial Fo},$$

which follows from Eqs. (2).

At $Fo > 0.15$ the effect of $dv/d\tau_e$, transmitted through capacitors C from network I-I to network II-II, becomes less than the effect of the decrease of potential due to the boundary conditions at point B_2 . Normally this corresponds to a change in the mass transfer potential due to thermal diffusion. Similar processes take place at point 8 of network I-I, but here, at small Fo , the "temperature" decreases owing to the expenditure of a certain amount of thermal energy on phase transitions.

§2. In order to check the experimental results we solved the boundary value problem analytically by means of a Laplace transformation of (2) with respect to the variable Fo .

The transforms were found by solving a system of ordinary differential equations in the variable X , after which the inverse transforms T and Θ were determined by means of the expansion theorem. After carrying out the necessary calculations, we obtained the results in the form of rapidly converging series. At the values $Ko^* = 0.333$, $Pn = 0.666$, and $Lu = 0.321$, corresponding to the experiment described above, we obtained the following final answer:

$$T(1; Fo) =$$

$$= 1 - 0.1264 \exp(-0.73Fo) - 1.293 \exp(-2.52Fo) +$$

$$+ 0.0812 \exp(-6.91Fo),$$

$$\Theta(0; Fo) =$$

$$= -1 + 1.555 \exp(-0.73Fo) - 0.3148 \exp(-2.52Fo) -$$

$$- 0.1878 \exp(-6.91Fo).$$

From these formulas we calculated the values of T and Θ for $Fo = 0.1, 0.3, 0.5$, and 1.0 (see Fig. 1). A comparison of the experimental results and the analytic calculation gave very good agreement.

§3. We now present the results of an analog simulation of the two-dimensional problem.

The $R_1 C_1$, $R_2 C_2$, and C networks, consisting of 52 cells each, were mounted on three textolite boards. The corresponding nodes of networks I and II and capacitors C were connected by means of plug-type connectors. The model was square-shaped with a rectangular notch cut out at one corner (see Fig. 2).

One series of experiments consisted in the sudden supplying of a constant electric potential $v^* = 100$ V at the boundary $A_1 B_1$ of network I-I and a potential $w^* = -100$ V at the boundary $A_2 B_2$ of network II-II with zero initial conditions. As the characteristic dimension we took the distance between points A_1 and E_1 (the length of eight cells). The criteria $Ko^* = 0.285$, $Pn = 0.571$, $Lu = 0.3065$ corresponded to the electrical network parameters $R_1 = 180$ k Ω ; $R_2 = 360$ k Ω ; $C_1 = 0.25 \cdot 8^2 = 16$ μ F; $C_2 = 0.5 \cdot 8^2 = 32$ μ F; $C = 0.1 \cdot 8^2 = 6.4$ μ F,

$$Fo = \frac{\tau_e}{R_1(C + C_1)} = \frac{\tau_e}{4.03},$$

where τ_e is the time of the transient process on the electrical model, sec.

The graphs of the dimensionless T and Θ versus Fo at various points of the model are similar in character to the graphs in Fig. 1. The distribution of T and Θ in the two-dimensional region at $Fo = 0.5$ is presented in Fig. 2. Curves of the family $T = \text{const}$ and $\Theta = \text{const}$ make it possible to construct the heat and mass flow lines. The greatest values of the temperature mass transfer potential gradients were observed at the instant indicated in the neighborhood of point D. It is clear from Fig. 2 that on part of the area the potential Θ assumed positive values owing to thermal diffusion. In solving two-dimensional problems of energy and mass transfer it is possible to use the networks of existing electro-integrators.

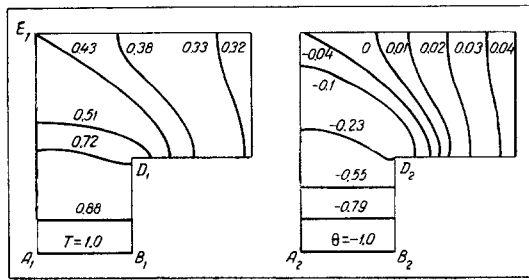


Fig. 2. Distribution of potentials T and Θ at the moment $Fo = 0.5$ in the two-dimensional problem.

§4. By using an electrical model it is possible to find simplified formulas for determining the potentials T and Θ for various problems. As an example, we present an approximate formula, in deriving which it is assumed that the distributed resistances and capacitances in networks I and II and the intermediate networks are replaced by lumped elements. This assumption is useful in studying the processes in thin-walled parts, and also at small Biot numbers Bi_q and Bi_m , when the coefficients α_q and α_m can be assumed constant and when the regular regime rapidly sets in.

We will consider the process of heat and mass transfer in an infinite plate with initial conditions $T(X; 0) = \Theta(X; 0) = 0$ and boundary conditions

$$\frac{\partial T(1; Fo)}{\partial X} + Bi_q [T(1; Fo) - 1] = 0,$$

$$\frac{\partial \Theta(1; Fo)}{\partial X} + Bi_m [\Theta(1; Fo) - 1] = 0.$$

Boundary conditions of the third kind were simulated by connecting additional resistances R_{b1} and R_{b2} at the surface of the model, such that $Bi_q = R_1 l_e / R_{b2}$, $Bi_m = R_2 l_c / R_{b2}$ (l_e is taken equal to 1). Solving the differential equations in the potentials T and Θ at the center of the board by an operational method, we obtain the formulas

$$T = 1 - \frac{b_m s_1 + Lu(1 + Ko^*)}{b_m(s_1 - s_2)} \exp(s_1 Fo) -$$

$$- \frac{b_m s_2 + Lu(1 + Ko^*)}{b_m(s_2 - s_1)} \exp(s_2 Fo),$$

$$\Theta = -1 + \frac{1 + s_1 b_q + Lu Pn(1 + Ko^*)}{b_q(s_1 - s_2)} \exp(s_1 Fo) +$$

$$+ \frac{1 + s_2 b_q + Lu Pn(1 + Ko^*)}{b_q(s_2 - s_1)} \exp(s_2 Fo),$$

$$b_q = 1 + \frac{1}{Bi_q}, \quad b_m = 1 + \frac{1}{Bi_m}, \quad \xi = Pn Ko^* + (1/Lu),$$

$$s_{1,2} = \frac{Lu}{2b_q} \left[-\left(\frac{b_q}{b_m} + \xi\right) \pm \sqrt{\left(\frac{b_q}{b_m} + \xi\right)^2 - \frac{4b_q}{Lu b_m}} \right].$$

§5. Analog simulation can be used to solve so-called contact problems, i. e., problems with discontinuous initial conditions. Physically, this means the contact

of two media along a certain boundary. At time $\tau = 0$, when contact occurs, the fluxes j_q and j_m at the contact points are infinitely large, but even after a very short interval of time both the temperatures and the mass transfer potentials in the contact zone are equalized and at $\tau > 0$ the fluxes vary continuously. Analytic solutions have been published for certain special cases of contact problems, but even in the simplest cases the solution leads to cumbersome calculations, while for complex configurations and arbitrary temperature and mass distributions an analytic solution is generally impossible. Therefore in our case we decided to solve contact problems by analog simulation.

We will consider, as an example, the solution of a one-dimensional contact problem. Let there be two infinite porous plates of identical thickness $l = 1$ with $Ko^* = 0.333$, $Lu = 1.025$, $Pn = 0.182$.

The initial values of T and Θ for the first plate are given by $T_{10} = 1$, $\Theta_{10} = -1$, and for the second by $T_{20} = 0$, $\Theta_{20} = 0$. At the boundary DE of the first plate the temperature and mass transfer potential are kept constant: $T(0; Fo) = 1$ and $\Theta(0; Fo) = -1$. At the surface AB of the second plate we have

$$\frac{\partial T}{\partial X} = 0, \quad \frac{\partial \Theta}{\partial X} = 0.$$

Let these two plates be brought into close contact at the instant $\tau = 0$. It is required to find the subsequent behavior of the functions $T(Fo)$ and $\Theta(Fo)$ at various points within the thickness of the plate.

The network of the electrical model used to solve this problem is shown in Fig. 3. To points D and E we supplied constant voltages $v^* = 100$ V and $w^* = -100$ V. At the moment $Fo = 0$ the contacts K_1 and K_2 are simultaneously closed. The lumped network parameters are equal to: $R_1 = 110$ k Ω , $R_2 = 105$ k Ω , $C = 0.25$ μ F, $C_1 = 0.5$ μ F, $C_2 = 1.0$ μ F. If τ_e is measured in seconds, then

$$Fo = \frac{\tau \cdot 1000}{(C_1 + C) R_1 l_e^2 \cdot 64}.$$

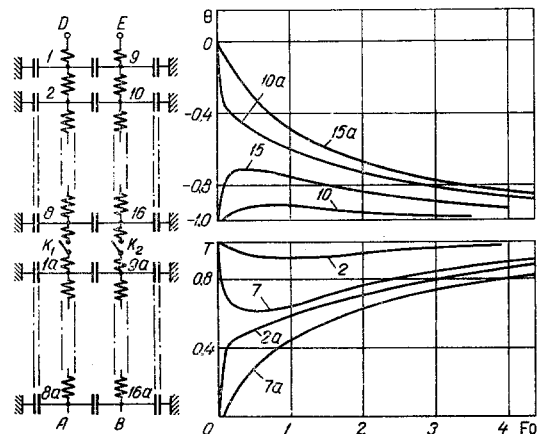


Fig. 3. Circuit of electrical model and graphs showing the variation of the parameters T , $\Theta = f(Fo)$ for various points of the medium in a contact problem.

The coefficient 64 in the denominator shows that there are 8 lumped capacitors and resistors per unit length of the model.

Graphs showing the variation of the relative temperature T at points 2, 7, 2a, and 7a and the potential Θ at points 10, 15, 10a, and 15a are presented in Fig. 3.

§6. These models can also be used to solve problems with boundary conditions of the second kind, which are of very great practical importance. For this purpose, to each point of the model simulating the surface of the medium it is necessary to supply certain currents I_1 and I_2 from high-voltage (v_c and w_c) potentiometers across sufficiently large resistances R_{b_1} and R_{b_2} . In this way the normal derivatives $\partial T/\partial N$ and $\partial \Theta/\partial N$, which depend on the surface coordinates, are specified at the boundaries of networks I and II.

If the criteria Ki_q and Ki_m depend on the potentials T and Θ at the surface of the model and on F_0 , then by means of a system of amplifiers it is possible to construct a circuit in which the voltages v_c and w_c depend in a certain way on the above-mentioned quantities.

NOTATION

τ is the time; a_q is the thermal diffusivity; ε is the ratio of the change of mass due to phase transformation in the neighborhood of some point to the total change of mass; r is the specific heat of phase transition; c_m is the isothermal mass capacity; c_q is the heat capacity; a_m is the mass diffusivity; δ_θ is the thermal gradient coefficient; l is the characteristic linear dimension; t^* , Θ^* are certain specific temperature and mass transfer potential drops; τ_e is the model time.

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